

Generalizations for reciprocal Fibonacci-Lucas sums of Brousseau*

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Abstract

We derive closed form expressions for finite and infinite Fibonacci-Lucas sums having products of Fibonacci or Lucas numbers in the denominator of the summand. Our results generalize and extend those obtained by pioneer Brother Alfred Brousseau and later researchers.

Contents

| | | |
|----------|--|----------|
| 1 | Introduction | 2 |
| 2 | Results: Generalizations | 5 |
| 2.1 | Telescoping summation identities | 5 |
| 2.2 | Sums with $F_{nk}F_{nk+nq} \cdots F_{nk+mnq}$ or $F_{nk}F_{nk+nq} \cdots F_{nk+(m-1)nq}F_{nk+(m+1)nq} \cdots F_{nk+2mnq}$ in the de- nominator | 8 |
| 2.3 | Sums with $L_{nk}L_{nk+nq} \cdots L_{nk+mnq}$ or $L_{nk}L_{nk+nq} \cdots L_{nk+mnq-nq}L_{nk+mnq+nq} \cdots L_{nk+2mnq}$ in the de- nominator | 15 |

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| | | |
|------|--|----|
| 2.4 | Sums with $F_{nk}F_{nk+nq} \cdots F_{nk+2mnq}$ in the denominator . . . | 18 |
| 2.5 | Sums with $L_{nk}L_{nk+nq} \cdots L_{nk+2mnq}$ in the denominator . . . | 20 |
| 2.6 | Sums with $F_{nk}F_{nk+2nq}F_{nk+4nq}F_{nk+6nq} \cdots F_{nk+2mnq}$ in the denominator | 20 |
| 2.7 | Sums with $L_{nk}L_{nk+2nq}L_{nk+4nq}L_{nk+6nq} \cdots L_{nk+2mnq}$ in the denominator | 22 |
| 2.8 | Sums with $F_{2nk}F_{2nk+2nq}F_{2nk+4nq}F_{2nk+6nq} \cdots F_{2nk+2mnq}$ in the denominator | 24 |
| 2.9 | Sums with $L_{2nk}L_{2nk+2nq}L_{2nk+4nq}L_{2nk+6nq} \cdots L_{2nk+2mnq}$ in the denominator | 26 |
| 2.10 | Sums with $F_{nk}^2F_{nk+nq}^2 \cdots F_{nk+mnq-nq}^2F_{nk+mnq}^2F_{nk+mnq+nq}^2 \cdots F_{nk+2mnq}^2$ or $F_{nk}^2F_{nk+nq}^2 \cdots F_{nk+mnq}^2$ in the denominator | 29 |
| 2.11 | Sums with $L_{nk}^2L_{nk+nq}^2 \cdots L_{nk+mnq-nq}^2L_{nk+mnq}^2L_{nk+mnq+nq}^2 \cdots L_{nk+2mnq}^2$ or $L_{nk}^2L_{nk+nq}^2 \cdots L_{nk+mnq}^2$ in the denominator | 33 |
| 2.12 | Sums with $F_{nk}F_{nk+np}F_{nk+nq}F_{nk+nq+np}F_{nk+2nq}F_{nk+2nq+np} \cdots F_{nk+mnq}F_{nk+mnq+np}$ in the denominator | 34 |
| 2.13 | Sums with $L_{nk}L_{nk+np}L_{nk+nq}L_{nk+nq+np}L_{nk+2nq}L_{nk+2nq+np} \cdots L_{nk+mnq}L_{nk+mnq+np}$ in the denominator | 36 |
| 2.14 | Evaluation of other sums | 37 |

1 Introduction

The Fibonacci numbers, F_n , and Lucas numbers, L_n , are defined, for $n \in \mathbb{N}_0$, as usual, through the recurrence relations $F_n = F_{n-1} + F_{n-2}$, with $F_0 = 0$, $F_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$, with $L_0 = 2$, $L_1 = 1$.

Our main aim in this paper is to derive closed form expressions for the following sums and their corresponding alternating versions, for positive integers m , n and q :

$$\sum_{k=1}^{\infty} \frac{L_{nk+nq}L_{nk+2nq} \cdots L_{nk+(m-1)nq}}{F_{nk}F_{nk+nq} \cdots F_{nk+mnq}}, \quad \sum_{k=1}^{\infty} \frac{F_{nk+nq}F_{nk+2nq} \cdots F_{nk+(m-1)nq}}{L_{nk}L_{nk+nq} \cdots L_{nk+mnq}}, \quad m > 1,$$

$$\sum_{k=1}^{\infty} \frac{1}{F_{nk}F_{nk+nq} \cdots F_{nk+mnq-nq}F_{nk+mnq+nq} \cdots F_{nk+mnq+2mnq}},$$

$$\sum_{k=1}^{\infty} \frac{1}{L_{nk}L_{nk+nq} \cdots L_{nk+mnq-nq}L_{nk+mnq+nq} \cdots L_{nk+mnq+2mnq}},$$

$$\begin{aligned}
& \sum_{k=1}^{\infty} \frac{L_{nk+mnq}}{F_{nk} F_{nk+nq} \cdots F_{nk+2mnq}}, \quad \sum_{k=1}^{\infty} \frac{F_{nk+mnq}}{L_{nk} L_{nk+nq} \cdots L_{nk+2mnq}}, \\
& \sum_{k=1}^{\infty} \frac{F_{2nk+mnq}}{F_{nk}^2 F_{nk+nq}^2 F_{nk+2nq}^2 \cdots F_{nk+mnq}^2}, \quad \sum_{k=1}^{\infty} \frac{F_{2nk+mnq}}{L_{nk}^2 L_{nk+nq}^2 L_{nk+2nq}^2 \cdots L_{nk+mnq}^2}, \\
& \sum_{k=1}^{\infty} \frac{L_{nk+mnq}}{F_{nk}^2 F_{nk+nq}^2 \cdots F_{nk+(m-1)nq}^2 F_{nk+mnq} F_{nk+(m+1)nq}^2 \cdots F_{nk+2mnq}^2}, \\
& \sum_{k=1}^{\infty} \frac{F_{nk+mnq}}{L_{nk}^2 L_{nk+nq}^2 \cdots L_{nk+(m-1)nq}^2 L_{nk+mnq} L_{nk+(m+1)nq}^2 \cdots L_{nk+2mnq}^2}, \\
& \sum_{k=1}^{\infty} \frac{F_{2nk+mnq} L_{nk+nq}^2 L_{nk+2nq}^2 \cdots L_{nk+(m-1)nq}^2}{F_{nk}^2 F_{nk+nq}^2 F_{nk+2nq}^2 \cdots F_{nk+mnq}^2}, \quad m > 1.
\end{aligned}$$

We require the following telescoping summation identities (see [1])

$$\sum_{k=1}^N [f(k) - f(k+q)] = \sum_{k=1}^q f(k) - \sum_{k=1}^q f(k+N), \quad \text{for } N \geq q \in \mathbb{N}_0 \quad (1.1)$$

and

$$\begin{aligned}
& \sum_{k=1}^N (-1)^{k-1} [f(k) + (-1)^{q-1} f(k+q)] \\
& = \sum_{k=1}^q (-1)^{k-1} f(k) + (-1)^{N-1} \sum_{k=1}^q (-1)^{k-1} f(k+N).
\end{aligned} \quad (1.2)$$

In general, infinite sums are evaluated using

$$\begin{aligned}
& \sum_{k=1}^{\infty} [f(k) - f(k+q)], \quad q \in \mathbb{N}_0 \\
& = \sum_{k=1}^q f(k) - \lim_{N \rightarrow \infty} \sum_{k=1}^q f(k+N)
\end{aligned} \quad (1.3)$$

and

$$\sum_{k=1}^{\infty} (-1)^{k-1} [f(k) \mp f(k+q)] = \sum_{k=1}^q (-1)^{k-1} f(k), \quad (1.4)$$

where the upper sign is taken if q is even and the lower if q is odd.

If $f(N)$ approaches zero as N approaches infinity, then we have, from (1.1) and (1.2), the useful identities

$$\sum_{k=1}^{\infty} [f(k) - f(k+q)] = \sum_{k=1}^q f(k), \quad q \in \mathbb{N}_0, \quad (1.5)$$

$$\sum_{k=1}^{\infty} (-1)^{k-1} [f(k) \mp f(k+q)] = \sum_{k=1}^q (-1)^{k-1} f(k), \quad (1.6)$$

where the upper sign applies if q is even and the lower if q is odd.

The golden ratio, having the numerical value of $(\sqrt{5} + 1)/2$, is denoted in this paper by ϕ .

We shall require the following identities (most of which can be found in the book by Vajda [6]):

$$L_v F_u = F_{u+v} + (-1)^v F_{u-v} \quad (1.7a)$$

$$F_v L_u = F_{u+v} - (-1)^v F_{u-v} \quad (1.7b)$$

$$2F_{u+v} = L_v F_u + L_u F_v \quad (1.8a)$$

$$(-1)^v 2F_{u-v} = F_u L_v - L_u F_v \quad (1.8b)$$

$$L_v L_u = L_{u+v} + (-1)^v L_{u-v} \quad (1.9a)$$

$$5F_v F_u = L_{u+v} - (-1)^v L_{u-v} \quad (1.9b)$$

$$(-1)^{u-1} (F_{v+u} F_{v-u}) = F_u^2 (F_{v+1} F_{v-1}) - F_v^2 (F_{u+1} F_{u-1}) \quad (1.10)$$

$$(-1)^t F_u F_v = F_{t+u} F_{t+v} - F_t F_{t+u+v} \quad (1.11a)$$

$$(-1)^{t+1} 5F_u F_v = L_{t+u} L_{t+v} - L_t L_{t+u+v} \quad (1.11b)$$

$$F_{u-v}F_{u+v} = F_u^2 + (-1)^{u+v-1}F_v^2 \quad (1.12a)$$

$$5F_{u-v}F_{u+v} = L_u^2 + (-1)^{u+v-1}L_v^2 \quad (1.12b)$$

$$F_vF_{2u+v+p} = F_{u+v+p}F_{u+v} + (-1)^{v+1}F_{u+p}F_u \quad (1.13a)$$

$$F_vL_{2u+v+p} = L_{u+v+p}F_{u+v} + (-1)^{v+1}L_{u+p}F_u \quad (1.13b)$$

The identities (1.12) and (1.13) were proved by Howard in [7].

The following limiting values are readily established using Binet's formula:

$$\lim_{N \rightarrow \infty} \frac{F_{N+m}}{F_{N+n}} = \phi^{m-n} = \lim_{N \rightarrow \infty} \frac{L_{N+m}}{L_{N+n}}, \quad (1.14a)$$

$$\lim_{N \rightarrow \infty} \frac{F_{N+m}}{L_{N+n}} = \frac{\phi^{m-n}}{\sqrt{5}} = \frac{1}{5} \lim_{N \rightarrow \infty} \frac{L_{N+m}}{F_{N+n}}. \quad (1.14b)$$

We shall adopt the following conventions for empty sums and empty products:

$$\sum_{k=1}^0 f(k) = 0, \quad \prod_{k=1}^0 f(k) = 1.$$

2 Results: Generalizations

2.1 Telescoping summation identities

Lemma 2.1. *If m, n, q and N are positive integers and $f(k)$ is a real sequence, then*

$$\begin{aligned} & \sum_{k=1}^N \left\{ [f(nk) - f(nk + mnq)] \prod_{j=1}^{m-1} f(nk + jnq) \right\} \\ &= \sum_{k=1}^q \left\{ \prod_{j=0}^{m-1} f(nk + jnq) \right\} - \sum_{k=1}^q \left\{ \prod_{j=0}^{m-1} f(nk + nN + jnq) \right\}. \end{aligned}$$

If the sequence $f(k)$ is convergent and we denote by f_∞ the limiting value of $f(nN)$ as N approaches infinity, we have

$$\begin{aligned} \sum_{k=1}^{\infty} \left\{ [f(nk) - f(nk + mnq)] \prod_{j=1}^{m-1} f(nk + jnq) \right\} \\ = \sum_{k=1}^q \left\{ \prod_{j=0}^{m-1} f(nk + jnq) \right\} - f_\infty^m q. \end{aligned} \quad (2.1)$$

Proof. We have

$$\begin{aligned} [f(nk) - f(nk + mnq)] \prod_{j=1}^{m-1} f(nk + jnq) \\ = f(nk) \prod_{j=1}^{m-1} f(nk + jnq) - f(nk + mnq) \prod_{j=1}^{m-1} f(nk + jnq) \\ = \prod_{j=0}^{m-1} f(nk + jnq) - \prod_{j=1}^m f(nk + jnq) \\ = \prod_{j=0}^{m-1} f(nk + jnq) - \prod_{j=0}^{m-1} f(nk + jnq + nq) \\ \equiv \prod_{j=0}^{m-1} f(nk + jnq) - \prod_{j=0}^{m-1} f(nk + jnq) \Big|_{k \rightarrow k+q}. \end{aligned} \quad (2.2)$$

The result follows by summing both sides of identity (2.2), using the identity (1.1) to perform the telescopic summation on the right hand side. \square

Lemma 2.2. *If $f(k)$ is a real sequence and m, n, q and N are positive integers such that q is even, then*

$$\begin{aligned} \sum_{k=1}^N (-1)^{k-1} \left\{ [f(nk) - f(nk + mnq)] \prod_{j=1}^{m-1} f(nk + jnq) \right\} \\ = \sum_{k=1}^q (-1)^{k-1} \left\{ \prod_{j=0}^{m-1} f(nk + jnq) \right\} \\ + (-1)^{N-1} \sum_{k=1}^q (-1)^{k-1} \left\{ \prod_{j=0}^{m-1} f(nk + nN + jnq) \right\}. \end{aligned}$$

Proof. Multiply through the identity (2.2) by $(-1)^{k-1}$ and use identity (1.2). \square

Lemma 2.3. *If $f(k)$ is a real sequence and m, n, q and N are positive integers such that q is odd, then*

$$\begin{aligned} & \sum_{k=1}^N (-1)^{k-1} \left\{ [f(nk) + f(nk + mnq)] \prod_{j=1}^{m-1} f(nk + jnq) \right\} \\ &= \sum_{k=1}^q (-1)^{k-1} \left\{ \prod_{j=0}^{m-1} f(nk + jnq) \right\} \\ &+ (-1)^{N-1} \sum_{k=1}^q (-1)^{k-1} \left\{ \prod_{j=0}^{m-1} f(nk + nN + jnq) \right\}. \end{aligned}$$

Proof. We have the identity

$$\begin{aligned} & [f(nk) + f(nk + mnq)] \prod_{j=1}^{m-1} f(nk + jnq) \\ &= \prod_{j=0}^{m-1} f(nk + jnq) + \prod_{j=0}^{m-1} f(nk + jnq + nq) \\ &\equiv \prod_{j=0}^{m-1} f(nk + jnq) + \prod_{j=0}^{m-1} f(nk + jnq) \Big|_{k \rightarrow k+q}, \end{aligned} \quad (2.3)$$

from which the result follows after multiplying through by $(-1)^{k-1}$ and summing over k , making use of the identity (1.2). \square

If the sequence $f(k)$ is convergent and $f(2Nn)$ and $f((2N-1)n)$ both have the same limiting value as N approaches infinity, we have

$$\begin{aligned} & \sum_{k=1}^{\infty} (-1)^{k-1} \left\{ [f(nk) \mp f(nk + mnq)] \prod_{j=1}^{m-1} f(nk + jnq) \right\} \\ &= \sum_{k=1}^q (-1)^{k-1} \left\{ \prod_{j=0}^{m-1} f(nk + jnq) \right\}, \end{aligned} \quad (2.4)$$

where the upper sign is taken if q is even and the lower if q is odd.

2.2 Sums with $F_{nk}F_{nk+nq}\cdots F_{nk+mnq}$ or $F_{nk}F_{nk+nq}\cdots F_{nk+(m-1)nq}F_{nk+(m+1)nq}\cdots F_{nk+2mnq}$ in the denominator

Theorem 2.4. *If m , n and q are positive integers, then*

$$\sum_{k=1}^{\infty} \left[(-1)^{nk-1} \frac{\prod_{j=1}^{m-1} L_{nk+jnq}}{\prod_{j=0}^m F_{nk+jnq}} \right] = \frac{q\sqrt{5^m}}{2F_{mnq}} - \frac{1}{2F_{mnq}} \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{L_{nk+jnq}}{F_{nk+jnq}},$$

so that

$$\sum_{k=1}^{\infty} \left[\frac{\prod_{j=1}^{m-1} L_{nk+jnq}}{\prod_{j=0}^m F_{nk+jnq}} \right] = \frac{1}{2F_{mnq}} \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{L_{nk+jnq}}{F_{nk+jnq}} - \frac{q\sqrt{5^m}}{2F_{mnq}}, \quad n \text{ even} \quad (2.5)$$

and

$$\sum_{k=1}^{\infty} \left[(-1)^{k-1} \frac{\prod_{j=1}^{m-1} L_{nk+jnq}}{\prod_{j=0}^m F_{nk+jnq}} \right] = \frac{q\sqrt{5^m}}{2F_{mnq}} - \frac{1}{2F_{mnq}} \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{L_{nk+jnq}}{F_{nk+jnq}}, \quad n \text{ odd}. \quad (2.6)$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{nk-1}}{F_{nk}F_{nk+nq}} = \frac{q\sqrt{5}}{2F_{nq}} - \frac{1}{2F_{nq}} \sum_{k=1}^q \frac{L_{nk}}{F_{nk}}. \quad (2.7)$$

Brousseau's result (equation (3) of [3], also rederived in various equivalent forms by other authors, see for example reference [4]) corresponds to setting $n = 1$ in (2.7), but with a different, but equivalent, form for the right hand side. Bruckman and Good's result (equation (19) of [5]) is also a special case of (2.7), corresponding to setting $q = 1$.

Theorem 2.5. *If m , n and q are integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{\prod_{j=1}^{m-1} L_{nk+jnq}}{\prod_{j=0}^m F_{nk+jnq}} \right] = \frac{1}{2F_{mnq}} \sum_{k=1}^q \left[(-1)^k \prod_{j=0}^{m-1} \frac{L_{nk+jnq}}{F_{nk+jnq}} \right].$$

In particular,

$$\sum_{k=1}^{\infty} \frac{1}{F_{nk} F_{nk+nq}} = \frac{1}{2F_{nq}} \sum_{k=1}^q \left[(-1)^k \frac{L_{nk}}{F_{nk}} \right], \quad n \text{ odd}, q \text{ even}, \quad (2.8)$$

which generalizes the result obtained by Rabinowitz (the second of equation (26) of [4]), the latter corresponding to the special case $n = 1$ in the identity (2.8), but with a different, but equivalent, form for the right hand side.

Proof of Theorem 2.4 and Theorem 2.5

Dividing through the identity (1.8b) by $F_u F_v$ and setting $u = nk + mnq$ and $v = nk$, the following identity is established for k, m, n and q positive integers:

$$\frac{(-1)^{nk-1} 2F_{mnq}}{F_{nk} F_{nk+mnq}} = \frac{L_{nk+mnq}}{F_{nk+mnq}} - \frac{L_{nk}}{F_{nk}}.$$

Using $f(k) = L_k/F_k$ in identity (2.1) we get the finite summation identity

$$\begin{aligned} 2F_{mnq} \sum_{k=1}^N \left[(-1)^{nk-1} \frac{\prod_{j=1}^{m-1} L_{nk+jnq}}{\prod_{j=0}^m F_{nk+jnq}} \right] \\ = \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{L_{nk+nN+jnq}}{F_{nk+nN+jnq}} \right] - \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{L_{nk+jnq}}{F_{nk+jnq}} \right], \end{aligned} \quad (2.9)$$

which yields Theorem 2.4 in the limit N approaches infinity. Theorem 2.5 is proved by using $f(k) = L_k/F_k$ in identity (2.4).

Theorem 2.6. *If m, n and q are positive odd integers, then*

$$\sum_{k=1}^{\infty} \left[\frac{1}{\prod_{j=0}^{m-1} F_{nk+njq} \prod_{j=m+1}^{2m} F_{nk+njq}} \right] = \frac{1}{L_{mnq}} \sum_{k=1}^q \left[\frac{1}{\prod_{j=0}^{2m-1} F_{nk+njq}} \right].$$

Below are a few explicit examples from Theorem 2.6:

At $m = 1$:

$$\sum_{k=1}^{\infty} \frac{1}{F_{nk} F_{nk+2nq}} = \frac{1}{L_{nq}} \sum_{k=1}^q \frac{1}{F_{nk} F_{nk+nq}}, \quad nq \text{ odd}. \quad (2.10)$$

At $(m, n, q) = (1, 1, 1)$ and $(m, n, q) = (1, 1, 3)$:

$$\sum_{k=1}^{\infty} \frac{1}{F_k F_{k+2}} = 1, \quad \sum_{k=1}^{\infty} \frac{1}{F_k F_{k+6}} = \frac{143}{960},$$

corresponding to Formulas (4) and (6) of Brousseau in reference [2].

At $(m, n, q) = (3, 1, 3)$:

$$\sum_{k=1}^{\infty} \frac{1}{F_k F_{k+3} F_{k+6} F_{k+12} F_{k+15} F_{k+18}} = \frac{938359017897442612}{5579104720519492358676480}.$$

At $(m, n, q) = (5, 3, 1)$:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{1}{F_{3k} F_{3k+3} F_{3k+6} F_{3k+9} F_{3k+12} F_{3k+18} F_{3k+21} F_{3k+24} F_{3k+27} F_{3k+30}} \\ = \frac{1}{13970032097862115517068710877593600}. \end{aligned}$$

Theorem 2.7. *If q , m and n are positive integers such that q is odd and nm is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+njq} \prod_{j=m+1}^{2m} F_{nk+njq}} \right] = \frac{1}{L_{mnq}} \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1} F_{nk+njq}} \right].$$

Examples from Theorem 2.7 include

At $m = 2$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{F_{nk} F_{nk+nq} F_{nk+3nq} F_{nk+4nq}} = \frac{1}{L_{2nq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{F_{nk} F_{nk+nq} F_{nk+2nq} F_{nk+3nq}}, \quad q \text{ odd}. \quad (2.11)$$

At $(m, n, q) = (1, 2, 1)$ and $(m, n, q) = (2, 1, 1)$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{F_{2k} F_{2k+4}} = \frac{1}{9}, \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{F_k F_{k+1} F_{k+3} F_{k+4}} = \frac{1}{18}.$$

At $(m, n, q) = (2, 6, 1)$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{F_{6k} F_{6k+6} F_{6k+18} F_{6k+24}} = \frac{1}{44444622716928}.$$

Proof of Theorem 2.6 and Theorem 2.7

With $v = mnq$ and $u = nk + mnq$ in identity (1.7a), the following identity is established:

$$\frac{L_{mnq}F_{nk+mnq}}{F_{nk}F_{nk+2mnq}} = \frac{1}{F_{nk}} + \frac{(-1)^{mnq}}{F_{nk+2mnq}}, \quad (2.12)$$

so that

$$\frac{L_{mnq}F_{nk+mnq}}{F_{nk}F_{nk+2mnq}} = \frac{1}{F_{nk}} - \frac{1}{F_{nk+2mnq}}, \quad mnq \text{ odd} \quad (2.13)$$

and

$$\frac{L_{mnq}F_{nk+mnq}}{F_{nk}F_{nk+2mnq}} = \frac{1}{F_{nk}} + \frac{1}{F_{nk+2mnq}}, \quad mnq \text{ even}. \quad (2.14)$$

If m , n and q are positive odd integers, then from (2.13) and using $f(k) = 1/F_k$ in Lemma 2.1 (with $m \rightarrow 2m$), we have the following definite summation identity

$$\begin{aligned} & \sum_{k=1}^N \left[\frac{1}{\prod_{j=0}^{m-1} F_{nk+jnq} \prod_{j=m+1}^{2m} F_{nk+jnq}} \right] \\ &= \frac{1}{L_{mnq}} \sum_{k=1}^q \left[\frac{1}{\prod_{j=0}^{2m-1} F_{nk+jnq}} \right] - \frac{1}{L_{mnq}} \sum_{k=1}^q \left[\frac{1}{\prod_{j=0}^{2m-1} F_{nk+nN+jnq}} \right], \end{aligned} \quad (2.15)$$

from which Theorem 2.6 follows as N approaches infinity.

If q is an odd positive integer and either m or n is even, then from (2.14) and Lemma 2.3 (with $m \rightarrow 2m$) we have the summation identity

$$\begin{aligned} & \sum_{k=1}^N \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+njq} \prod_{j=m+1}^{2m} F_{nk+njq}} \right] \\ &= \frac{1}{L_{mnq}} \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1} F_{nk+njq}} \right] + \frac{(-1)^{N-1}}{L_{mnq}} \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1} F_{nk+nN+njq}} \right], \end{aligned} \quad (2.16)$$

from which Theorem 2.7 follows in the limit that N approaches infinity.

Theorem 2.8. *If m , n and q are positive integers such that q is odd, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+mnq} \prod_{j=1}^{m-1} L_{nk+jnq}}{\prod_{j=0}^m F_{nk+jnq}} = \frac{1}{2} \sum_{k=1}^q (-1)^{k-1} \prod_{j=0}^{m-1} \frac{L_{nk+jnq}}{F_{nk+jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+nq}}{F_{nk} F_{nk+nq}} = \frac{1}{2} \sum_{k=1}^q (-1)^{k-1} \frac{L_{nk}}{F_{nk}}, \quad q \text{ odd}. \quad (2.17)$$

Corollary 2.9. *If m , n and q are positive integers such that q is odd, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{nk+mnq}^2 \prod_{j=1}^{m-1} L_{nk+jnq} \prod_{j=m+1}^{2m-1} L_{nk+jnq}}{\prod_{j=0}^{m-1} F_{nk+jnq} \prod_{j=m+1}^{2m} F_{nk+jnq}} = \frac{1}{2} \sum_{k=1}^q (-1)^{k-1} \prod_{j=0}^{2m-1} \frac{L_{nk+jnq}}{F_{nk+jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{nk+nq}^2}{F_{nk} F_{nk+2nq}} = \frac{1}{2} \sum_{k=1}^q (-1)^{k-1} \frac{L_{nk} L_{nk+nq}}{F_{nk} F_{nk+nq}}, \quad q \text{ odd}. \quad (2.18)$$

Proof of Theorem 2.8 and Corollary 2.9

Dividing through the identity (1.8a) by $F_u F_v$ and setting $u = nk + mnq$ and $v = nk$ we have the identity

$$2 \frac{F_{2nk+mnq}}{F_{nk} F_{nk+mnq}} = \frac{L_{nk+mnq}}{F_{nk+mnq}} + \frac{L_{nk}}{F_{nk}}. \quad (2.19)$$

If q is odd, then from (2.19) and with $f(k) = L_k/F_k$ in Lemma 2.3 we have

$$\begin{aligned} 2 \sum_{k=1}^N \frac{(-1)^{k-1} F_{2nk+mnq} \prod_{j=1}^{m-1} L_{nk+jnq}}{\prod_{j=0}^m F_{nk+jnq}} \\ = \sum_{k=1}^q (-1)^{k-1} \prod_{j=0}^{m-1} \frac{L_{nk+jnq}}{F_{nk+jnq}} + (-1)^{N-1} \sum_{k=1}^q (-1)^{k-1} \prod_{j=0}^{m-1} \frac{L_{nk+nN+jnq}}{F_{nk+nN+jnq}}, \end{aligned} \quad (2.20)$$

from which Theorem (2.8) follows in the limit as N approaches infinity. Corollary (2.9) is obtained by specifically requiring m to be even in Theorem (2.8).

Theorem 2.10. *If m, n, q and p are positive integers, then*

$$\begin{aligned} \sum_{k=1}^{\infty} \left\{ \frac{(-1)^{nk-1} \prod_{j=1}^{m-1} F_{nk+jnq+np}}{\prod_{j=0}^m F_{nk+jnq}} \right\} \\ = \frac{\phi^{mnp} q}{F_{mnq} F_{np}} - \frac{1}{F_{mnq} F_{np}} \sum_{k=1}^q \left\{ \prod_{j=0}^{m-1} \frac{F_{nk+jnq+np}}{F_{nk+jnq}} \right\}. \end{aligned}$$

In particular we have

$$\sum_{k=1}^{\infty} \left\{ \frac{(-1)^{nk-1} \prod_{j=m+1}^{p+m-1} F_{nk+jn}}{\prod_{j=0}^p F_{nk+jn}} \right\} = \frac{\phi^{mnp}}{F_{mn} F_{pn}} - \frac{1}{F_{mn} F_{pn}} \prod_{j=0}^{m-1} \frac{F_{jn+np+n}}{F_{jn+n}} \quad (2.21)$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{nk-1}}{F_{nk} F_{nk+nq}} = \frac{\phi^n q}{F_{nq} F_n} - \frac{1}{F_{nq} F_n} \sum_{k=1}^q \frac{F_{nk+n}}{F_{nk}}. \quad (2.22)$$

Observe that identity (2.22) is equivalent to identity (2.7) but with a different form for the right hand side. Since $2\phi = \sqrt{5} + 1$, $F_{n-1} + F_{n+1} = L_n$ and $\phi^n = \phi F_n + F_{n-1}$, both identities can be combined to yield the following interesting summation identity which is valid for all non-zero integers n and non-negative integers q :

$$\frac{1}{F_n} \sum_{k=1}^q \left[\frac{F_{nk+n}}{F_{nk}} \right] - \frac{1}{2} \sum_{k=1}^q \left[\frac{L_{nk}}{F_{nk}} \right] = \frac{q}{2} \frac{L_n}{F_n}. \quad (2.23)$$

Theorem 2.11. *If m, n, q and p are positive integers such that n is odd and q is even, then*

$$\begin{aligned} \sum_{k=1}^{\infty} \left\{ \frac{\prod_{j=1}^{m-1} F_{nk+jnq+np}}{\prod_{j=0}^m F_{nk+jnq}} \right\} \\ = \frac{1}{F_{mnq} F_{np}} \sum_{k=1}^q \left\{ (-1)^k \prod_{j=0}^{m-1} \frac{F_{nk+jnq+np}}{F_{nk+jnq}} \right\}. \end{aligned}$$

In particular,

$$\sum_{k=1}^{\infty} \frac{1}{F_{nk} F_{nk+nq}} = \frac{1}{F_n F_{nq}} \sum_{k=1}^q \left[(-1)^k \frac{F_{nk+n}}{F_{nk}} \right], \quad n \text{ odd, } q \text{ even}. \quad (2.24)$$

From identity (2.8) and identity (2.24) we have the interesting result

$$\frac{1}{2} \sum_{k=1}^q \left[(-1)^k \frac{L_{nk}}{F_{nk}} \right] = \frac{1}{F_n} \sum_{k=1}^q \left[(-1)^k \frac{F_{nk+n}}{F_{nk}} \right], \quad q \text{ even}. \quad (2.25)$$

Proof of Theorem 2.10 and Theorem 2.11

Dividing through identity (1.11a) by $F_{t+u}F_t$ and choosing $t = nk$, $u = mnq$ and $v = np$ we obtain the identity:

$$\frac{(-1)^{nk-1} F_{mnq} F_{np}}{F_{nk} F_{nk+mnq}} = \frac{F_{nk+mnq+np}}{F_{nk+mnq}} - \frac{F_{nk+np}}{F_{nk}}. \quad (2.26)$$

With $f(k) = F_{k+np}/F_k$ in Lemma 2.1 and using the identity (2.26), we have the finite summation identity

$$\begin{aligned} & F_{mnq} F_{np} \sum_{k=1}^N \left\{ \frac{(-1)^{nk-1} \prod_{j=1}^{m-1} F_{nk+jnq+np}}{\prod_{j=0}^m F_{nk+jnq}} \right\} \\ &= \sum_{k=1}^q \left\{ \prod_{j=0}^{m-1} \frac{F_{nk+nN+jnq+np}}{F_{nk+nN+jnq}} \right\} - \sum_{k=1}^q \left\{ \prod_{j=0}^{m-1} \frac{F_{nk+jnq+np}}{F_{nk+jnq}} \right\}, \end{aligned} \quad (2.27)$$

from which Theorem 2.10 follows in the limit as N approaches infinity.

Using $f(k) = F_{k+np}/F_k$ in Lemma 2.2 gives

$$\begin{aligned} & F_{mnq} F_{np} \sum_{k=1}^N \left\{ \frac{\prod_{j=1}^{m-1} F_{nk+jnq+np}}{\prod_{j=0}^m F_{nk+jnq}} \right\} \\ &= (-1)^N \sum_{k=1}^q \left\{ (-1)^{k-1} \prod_{j=0}^{m-1} \frac{F_{nk+nN+jnq+np}}{F_{nk+nN+jnq}} \right\} \\ &\quad - \sum_{k=1}^q \left\{ (-1)^{k-1} \prod_{j=0}^{m-1} \frac{F_{nk+jnq+np}}{F_{nk+jnq}} \right\}, \end{aligned} \quad (2.28)$$

from which Theorem 2.11 follows.

2.3 Sums with $L_{nk}L_{nk+nq} \cdots L_{nk+mnq}$ or $L_{nk}L_{nk+nq} \cdots L_{nk+mnq-nq}L_{nk+mnq+nq} \cdots L_{nk+2mnq}$ in the denominator

The derivations here proceed in the same fashion as in the previous section. The theorems will therefore be stated without proof. The analogous identity to (2.9) is

$$\begin{aligned} 2F_{mnq} \sum_{k=1}^N \left[(-1)^{nk-1} \frac{\prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^m L_{nk+jnq}} \right] \\ = \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{F_{nk+jnq}}{L_{nk+jnq}} \right] - \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{F_{nk+nN+jnq}}{L_{nk+nN+jnq}} \right]. \end{aligned} \quad (2.29)$$

Theorem 2.12. *If m , n and q are positive integers, then*

$$\sum_{k=1}^{\infty} \left[(-1)^{nk-1} \frac{\prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^m L_{nk+jnq}} \right] = \frac{1}{2F_{mnq}} \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{F_{nk+jnq}}{L_{nk+jnq}} - \frac{q}{2F_{mnq}\sqrt{5^m}},$$

so that

$$\sum_{k=1}^{\infty} \left[\frac{\prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^m L_{nk+jnq}} \right] = \frac{q}{2F_{mnq}\sqrt{5^m}} - \frac{1}{2F_{mnq}} \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{F_{nk+jnq}}{L_{nk+jnq}}, \quad n \text{ even} \quad (2.30)$$

and

$$\sum_{k=1}^{\infty} \left[(-1)^{k-1} \frac{\prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^m L_{nk+jnq}} \right] = \frac{1}{2F_{mnq}} \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{F_{nk+jnq}}{L_{nk+jnq}} - \frac{q}{2F_{mnq}\sqrt{5^m}}, \quad n \text{ odd}. \quad (2.31)$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{nk-1}}{L_{nk}L_{nk+nq}} = \frac{1}{2F_{nq}} \sum_{k=1}^q \frac{F_{nk}}{L_{nk}} - \frac{q}{2F_{nq}\sqrt{5}}, \quad n, q \in \mathbb{Z}^+. \quad (2.32)$$

The case $n = 3, q = 1$ in (2.32) was mentioned by Brousseau (equation (14) of [2]).

Theorem 2.13. *If m , n and q are integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{\prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^m L_{nk+jnq}} \right] = \frac{1}{2F_{mnq}} \sum_{k=1}^q \left[(-1)^{k-1} \prod_{j=0}^{m-1} \frac{F_{nk+jnq}}{L_{nk+jnq}} \right].$$

In particular,

$$\sum_{k=1}^{\infty} \frac{1}{L_{nk} L_{nk+nq}} = \frac{1}{2F_{nq}} \sum_{k=1}^q \left[(-1)^{k-1} \frac{F_{nk}}{L_{nk}} \right], \quad n \text{ odd}, q \text{ even}. \quad (2.33)$$

Theorem 2.14. *If m , n and q are positive odd integers, then*

$$\sum_{k=1}^{\infty} \left[\frac{1}{\prod_{j=0}^{m-1} L_{nk+njq} \prod_{j=m+1}^{2m} L_{nk+njq}} \right] = \frac{1}{L_{mnq}} \sum_{k=1}^q \left[\frac{1}{\prod_{j=0}^{2m-1} L_{nk+njq}} \right].$$

Theorem 2.15. *If q , m and n are positive integers such that q is odd and nm is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+njq} \prod_{j=m+1}^{2m} L_{nk+njq}} \right] = \frac{1}{L_{mnq}} \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1} L_{nk+njq}} \right].$$

Analogous identity to identity (2.20) is

$$\begin{aligned} 2 \sum_{k=1}^N \frac{(-1)^{k-1} F_{2nk+mnq} \prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^m L_{nk+jnq}} \\ = \sum_{k=1}^q (-1)^{k-1} \prod_{j=0}^{m-1} \frac{F_{nk+jnq}}{L_{nk+jnq}} + (-1)^{N-1} \sum_{k=1}^q (-1)^{k-1} \prod_{j=0}^{m-1} \frac{F_{nk+nN+jnq}}{L_{nk+nN+jnq}}, \end{aligned} \quad (2.34)$$

from which we get the following theorem in the limit as N approaches infinity.

Theorem 2.16. *If m , n and q are positive integers such that q is odd, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+mnq} \prod_{j=1}^{m-1} F_{nk+jnq}}{\prod_{j=0}^m L_{nk+jnq}} = \frac{1}{2} \sum_{k=1}^q (-1)^{k-1} \prod_{j=0}^{m-1} \frac{F_{nk+jnq}}{L_{nk+jnq}}.$$

Corollary 2.17. *If m, n and q are positive integers such that q is odd, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{nk+mnq}^2 \prod_{j=1}^{m-1} F_{nk+jnq} \prod_{j=m+1}^{2m-1} F_{nk+jnq}}{\prod_{j=0}^{m-1} L_{nk+jnq} \prod_{j=m+1}^{2m} L_{nk+jnq}} = \frac{1}{2} \sum_{k=1}^q (-1)^{k-1} \prod_{j=0}^{2m-1} \frac{F_{nk+jnq}}{L_{nk+jnq}}.$$

Corresponding to identity (2.26) of section 2.2 we have (from identity (1.11b))

$$\frac{(-1)^{nk-1} 5F_{mnq} F_{np}}{L_{nk} L_{nk+mnq}} = -\frac{L_{nk+mnq+np}}{L_{nk+mnq}} + \frac{L_{nk+np}}{L_{nk}}, \quad (2.35)$$

leading to the summation identities

$$\begin{aligned} & 5F_{mnq} F_{np} \sum_{k=1}^N \left\{ \frac{(-1)^{nk-1} \prod_{j=1}^{m-1} L_{nk+jnq+np}}{\prod_{j=0}^m L_{nk+jnq}} \right\} \\ &= - \sum_{k=1}^q \left\{ \prod_{j=0}^{m-1} \frac{L_{nk+nN+jnq+np}}{L_{nk+nN+jnq}} \right\} + \sum_{k=1}^q \left\{ \prod_{j=0}^{m-1} \frac{L_{nk+jnq+np}}{L_{nk+jnq}} \right\} \end{aligned} \quad (2.36)$$

and

$$\begin{aligned} & 5F_{mnq} F_{np} \sum_{k=1}^N \left\{ \frac{\prod_{j=1}^{m-1} L_{nk+jnq+np}}{\prod_{j=0}^m L_{nk+jnq}} \right\} \\ &= (-1)^{N-1} \sum_{k=1}^q \left\{ (-1)^{k-1} \prod_{j=0}^{m-1} \frac{L_{nk+nN+jnq+np}}{L_{nk+nN+jnq}} \right\} \\ &+ \sum_{k=1}^q \left\{ (-1)^{k-1} \prod_{j=0}^{m-1} \frac{L_{nk+jnq+np}}{L_{nk+jnq}} \right\}, \end{aligned} \quad (2.37)$$

from which Theorem 2.18 and Theorem 2.19 follow.

Theorem 2.18. *If m, n, q and p are positive integers, then*

$$\begin{aligned} & \sum_{k=1}^{\infty} \left\{ \frac{(-1)^{nk-1} \prod_{j=1}^{m-1} L_{nk+jnq+np}}{\prod_{j=0}^m L_{nk+jnq}} \right\} \\ &= -\frac{\phi^{mnp} q}{5F_{mnq} F_{np}} + \frac{1}{5F_{mnq} F_{np}} \sum_{k=1}^q \left\{ \prod_{j=0}^{m-1} \frac{L_{nk+jnq+np}}{L_{nk+jnq}} \right\}. \end{aligned}$$

In particular we have

$$\sum_{k=1}^{\infty} \left\{ \frac{(-1)^{nk-1} \prod_{j=m+1}^{p+m-1} L_{nk+jn}}{\prod_{j=0}^p L_{nk+jn}} \right\} = -\frac{\phi^{mnp}}{5F_{mn}F_{pn}} + \frac{1}{5F_{mn}F_{pn}} \prod_{j=0}^{m-1} \frac{L_{jn+np+n}}{L_{jn+n}} \quad (2.38)$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{nk-1}}{L_{nk}L_{nk+nq}} = -\frac{\phi^n q}{5F_{nq}F_n} + \frac{1}{5F_{nq}F_n} \sum_{k=1}^q \frac{L_{nk+n}}{L_{nk}}. \quad (2.39)$$

Theorem 2.19. *If m, n, q and p are positive integers such that n is odd and q is even, then*

$$\begin{aligned} \sum_{k=1}^{\infty} \left\{ \frac{\prod_{j=1}^{m-1} L_{nk+jnq+np}}{\prod_{j=0}^m L_{nk+jnq}} \right\} \\ = \frac{1}{5F_{mnq}F_{np}} \sum_{k=1}^q \left\{ (-1)^{k-1} \prod_{j=0}^{m-1} \frac{L_{nk+jnq+np}}{L_{nk+jnq}} \right\}. \end{aligned}$$

In particular,

$$\sum_{k=1}^{\infty} \frac{1}{L_{nk}L_{nk+nq}} = \frac{1}{5F_nF_{nq}} \sum_{k=1}^q \left[(-1)^{k-1} \frac{L_{nk+n}}{L_{nk}} \right], \quad n \text{ odd, } q \text{ even}. \quad (2.40)$$

From identity (2.33) and identity (2.40) we have

$$\frac{1}{2} \sum_{k=1}^q \left[(-1)^{k-1} \frac{F_{nk}}{L_{nk}} \right] = \frac{1}{5F_n} \sum_{k=1}^q \left[(-1)^{k-1} \frac{L_{nk+n}}{L_{nk}} \right], \quad q \text{ even}. \quad (2.41)$$

2.4 Sums with $F_{nk}F_{nk+nq} \cdots F_{nk+2mnq}$ in the denominator

The results in this section are obtained from identity (1.7b). We have

$$\frac{F_v L_u}{F_{u-v} F_{u+v}} = \frac{1}{F_{u-v}} - \frac{(-1)^v}{F_{u+v}},$$

from which, by setting $v = mnq$ and $u = nk + mnq$, we get

$$\frac{F_{mnq} L_{nk+mnq}}{F_{nk} F_{nk+2mnq}} = \frac{1}{F_{nk}} - \frac{(-1)^{mnq}}{F_{nk+2mnq}}, \quad (2.42)$$

so that

$$\frac{F_{mnq}L_{nk+mnq}}{F_{nk}F_{nk+2mnq}} = \frac{1}{F_{nk}} - \frac{1}{F_{nk+2mnq}}, \quad mnq \text{ even} \quad (2.43)$$

and

$$\frac{F_{mnq}L_{nk+mnq}}{F_{nk}F_{nk+2mnq}} = \frac{1}{F_{nk}} + \frac{1}{F_{nk+2mnq}}, \quad mnq \text{ odd}. \quad (2.44)$$

The derivations then proceed as in the previous sections.

Theorem 2.20. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{L_{nk+mnq}}{\prod_{j=0}^{2m} F_{nk+njq}} \right] = \frac{1}{F_{mnq}} \sum_{k=1}^q \left[\frac{1}{\prod_{j=0}^{2m-1} F_{nk+njq}} \right]$$

Examples from Theorem 2.20 include:

$$\text{At } (m, n, q) = (1, 2, 1) \text{ and } (m, n, q) = (1, 1, 2):$$

$$\sum_{k=1}^{\infty} \frac{L_{2k+2}}{F_{2k}F_{2k+2}F_{2k+4}} = \frac{1}{3}, \quad \sum_{k=1}^{\infty} \frac{L_{k+2}}{F_kF_{k+2}F_{k+4}} = \frac{5}{6}. \quad (2.45)$$

The first of the identities in (2.45) was also derived in [10] (equation 3.7).

At $(m, n, q) = (2, 7, 1)$:

$$\sum_{k=1}^{\infty} \frac{L_{7k+14}}{F_{7k}F_{7k+7}F_{7k+14}F_{7k+21}F_{7k+28}} = \frac{1}{6427623373464462}.$$

Theorem 2.21. *If m, n and q are positive integers such that q is even or mnq is odd, then*

$$\sum_{k=1}^{\infty} \left[\frac{(-1)^{k-1}L_{nk+mnq}}{\prod_{j=0}^{2m} F_{nk+njq}} \right] = \frac{1}{F_{mnq}} \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1} F_{nk+njq}} \right]$$

Examples from Theorem 2.21 include:

At $(m, n, q) = (1, 1, 2)$ and $(m, n, q) = (1, 3, 2)$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}L_{k+2}}{F_kF_{k+2}F_{k+4}} = \frac{1}{6}, \quad \sum_{k=1}^{\infty} \frac{(-1)^{k-1}L_{3k+6}}{F_{3k}F_{3k+6}F_{3k+12}} = \frac{271}{156672}.$$

At $(m, n, q) = (2, 4, 2)$:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}L_{4k+16}}{F_{4k}F_{4k+8}F_{4k+16}F_{4k+24}F_{4k+32}} = \frac{177072540680427}{166704475185956548320480}.$$

2.5 Sums with $L_{nk}L_{nk+nq} \cdots L_{nk+2mnq}$ in the denominator

The results here follow from the identity (1.9b).

Theorem 2.22. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{F_{nk+mnq}}{\prod_{j=0}^{2m} L_{nk+jnq}} \right] = \frac{1}{5F_{mnq}} \sum_{k=1}^q \left[\prod_{j=0}^{2m-1} \frac{1}{L_{nk+jnq}} \right].$$

Theorem 2.23. *If m, n and q are positive integers such that q is even or mnq is odd, then*

$$\sum_{k=1}^{\infty} (-1)^{k-1} \left[\frac{F_{nk+mnq}}{\prod_{j=0}^{2m} L_{nk+jnq}} \right] = \frac{1}{5F_{mnq}} \sum_{k=1}^q \left[(-1)^{k-1} \prod_{j=0}^{2m-1} \frac{1}{L_{nk+jnq}} \right].$$

2.6 Sums with $F_{nk}F_{nk+2nq}F_{nk+4nq}F_{nk+6nq} \cdots F_{nk+2mnq}$ in the denominator

Theorem 2.24. *If m, n and q are positive odd integers, then*

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1} F_{nk+mnq}}{\prod_{j=0}^m F_{nk+2jnq}} = \frac{1}{L_{mnq}} \sum_{k=1}^{2q} \frac{(\pm 1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{F_{nk+nq}}{F_{nk}F_{nk+2nq}} = \frac{1}{L_{nq}} \sum_{k=1}^{2q} \frac{1}{F_{nk}}, \quad nq \text{ odd}. \quad (2.46)$$

Proof. From identity (2.13) and with $f(k) = 1/F_k$ (and $q \rightarrow 2q$) in Lemma 2.1 we have the finite summation identity

$$\begin{aligned} L_{mnq} \sum_{k=1}^N \frac{F_{nk+mnq}}{\prod_{j=0}^m F_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} F_{nk+2jnq}} \quad (mnq \text{ odd}) \\ &\quad - \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} F_{nk+nN+2jnq}}. \end{aligned} \quad (2.47)$$

From identity (2.13) with m , n and q positive odd intergers and with $f(k) = 1/F_k$ (and $q \rightarrow 2q$) in Lemma 2.2 we have the alternating finite summation identity

$$\begin{aligned} L_{mnq} \sum_{k=1}^N \frac{(-1)^{k-1} F_{nk+mnq}}{\prod_{j=0}^m F_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+2jnq}} \\ &+ (-1)^{N-1} \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+nN+2jnq}} \end{aligned} \quad (mnq \text{ odd}) \quad (2.48)$$

Theorem 2.24 follows from identities (2.47) and (2.48) in the limit as N approaches infinity. \square

Theorem 2.25. *If m , n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1} L_{nk+mnq}}{\prod_{j=0}^m F_{nk+2jnq}} = \frac{1}{F_{mnq}} \sum_{k=1}^{2q} \frac{(\pm 1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{L_{nk+nq}}{F_{nk} F_{nk+2nq}} = \frac{1}{F_{nq}} \sum_{k=1}^{2q} \frac{1}{F_{nk}}, \quad nq \text{ even}. \quad (2.49)$$

Proof. As in Theorem 2.24, with the identity (2.43), with mnq even.

The corresponding finite summation identities are

$$\begin{aligned} F_{mnq} \sum_{k=1}^N \frac{L_{nk+mnq}}{\prod_{j=0}^m F_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} F_{nk+2jnq}} \\ &- \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} F_{nk+nN+2jnq}} \end{aligned} \quad (mnq \text{ even}) \quad (2.50)$$

and

$$\begin{aligned}
F_{mnq} \sum_{k=1}^N \frac{(-1)^{k-1} L_{nk+mnq}}{\prod_{j=0}^m F_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+2jnq}} \quad (mnq \text{ even}) \\
&+ (-1)^{N-1} \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+nN+2jnq}}.
\end{aligned} \tag{2.51}$$

□

2.7 Sums with $L_{nk}L_{nk+2nq}L_{nk+4nq}L_{nk+6nq} \cdots L_{nk+2mnq}$ in the denominator

In this section we state the Lucas versions of the results given in section 2.6.

Here the basic identities (from identities (1.9)) are:

$$\frac{L_{mnq}L_{nk+mnq}}{L_{nk}L_{nk+2mnq}} = \frac{1}{L_{nk}} - \frac{1}{L_{nk+2mnq}}, \quad mnq \text{ odd} \tag{2.52}$$

$$\frac{5F_{mnq}F_{nk+mnq}}{L_{nk}L_{nk+2mnq}} = \frac{1}{L_{nk}} - \frac{1}{L_{nk+2mnq}}, \quad mnq \text{ even}. \tag{2.53}$$

Theorem 2.26. *If m , n and q are positive odd integers, then*

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1} L_{nk+mnq}}{\prod_{j=0}^m L_{nk+2jnq}} = \frac{1}{L_{mnq}} \sum_{k=1}^{2q} \frac{(\pm 1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{L_{nk+nq}}{L_{nk}L_{nk+2nq}} = \frac{1}{L_{nq}} \sum_{k=1}^{2q} \frac{1}{L_{nk}}, \quad nq \text{ odd}. \tag{2.54}$$

Theorem 2.27. *If m , n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \frac{(\pm 1)^{k-1} F_{nk+mnq}}{\prod_{j=0}^m L_{nk+2jnq}} = \frac{1}{5F_{mnq}} \sum_{k=1}^{2q} \frac{(\pm 1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{F_{nk+nq}}{L_{nk}L_{nk+2nq}} = \frac{1}{5F_{nq}} \sum_{k=1}^{2q} \frac{1}{F_{nk}}, \quad nq \text{ even}. \quad (2.55)$$

We have the following finite summation identities:

$$\begin{aligned} L_{mnq} \sum_{k=1}^N \frac{L_{nk+mnq}}{\prod_{j=0}^m L_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} L_{nk+2jnq}} \quad (mnq \text{ odd}) \\ &\quad - \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} L_{nk+nN+2jnq}}. \end{aligned} \quad (2.56)$$

$$\begin{aligned} L_{mnq} \sum_{k=1}^N \frac{(-1)^{k-1} L_{nk+mnq}}{\prod_{j=0}^m L_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+2jnq}} \quad (mnq \text{ odd}) \\ &\quad + (-1)^{N-1} \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+nN+2jnq}}. \end{aligned} \quad (2.57)$$

$$\begin{aligned} 5F_{mnq} \sum_{k=1}^N \frac{F_{nk+mnq}}{\prod_{j=0}^m L_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} L_{nk+2jnq}} \quad (mnq \text{ even}) \\ &\quad - \sum_{k=1}^{2q} \frac{1}{\prod_{j=0}^{m-1} L_{nk+nN+2jnq}}. \end{aligned} \quad (2.58)$$

$$\begin{aligned} 5F_{mnq} \sum_{k=1}^N \frac{(-1)^{k-1} F_{nk+mnq}}{\prod_{j=0}^m L_{nk+2jnq}} &= \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+2jnq}} \quad (mnq \text{ even}) \\ &\quad + (-1)^{N-1} \sum_{k=1}^{2q} \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+nN+2jnq}}. \end{aligned} \quad (2.59)$$

2.8 Sums with $F_{2nk}F_{2nk+2nq}F_{2nk+4nq}F_{2nk+6nq}\cdots F_{2nk+2mnq}$ in the denominator

Theorem 2.28. *If m , n and q are positive odd integers, then*

$$\sum_{k=1}^{\infty} \frac{F_{2nk+mnq}}{\prod_{j=0}^m F_{2nk+2jnq}} = \frac{1}{L_{mnq}} \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{2nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{F_{2nk+nq}}{F_{2nk}F_{2nk+2nq}} = \frac{1}{L_{nq}} \sum_{k=1}^q \frac{1}{F_{2nk}}, \quad (2.60)$$

$$\sum_{k=1}^{\infty} \frac{F_{2nk+3nq}}{F_{2nk}F_{2nk+2nq}F_{2nk+4nq}F_{2nk+6nq}} = \frac{1}{L_{3nq}} \sum_{k=1}^q \frac{1}{F_{2nk}F_{2nk+2nq}F_{2nk+4nq}}. \quad (2.61)$$

Identity (18) of reference [9] with $n = 0$ in their notation corresponds to setting $q = 1$ in identity (2.60).

Proof. From identity (1.7a) with $v = mnq$ and $u = 2nk + mnq$ comes the identity

$$\frac{L_{mnq}F_{2nk+mnq}}{F_{2nk}F_{2nk+2mnq}} = \frac{1}{F_{2nk}} - \frac{1}{F_{2nk+2mnq}}, \quad mnq \text{ odd}. \quad (2.62)$$

From identity (2.62) and Lemma 2.1 with $f(k) = 1/F_{2k}$ we have the finite summation identity:

$$\begin{aligned} L_{mnq} \sum_{k=1}^N \frac{F_{2nk+mnq}}{\prod_{j=0}^m F_{2nk+2jnq}} &= \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{2nk+2jnq}} \\ &\quad - \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{2nk+2nN+2jnq}}, \end{aligned} \quad (2.63)$$

from which Theorem 2.28 follows in the limit as N approaches infinity. \square

Theorem 2.29. *If m , n and q are positive integers such that q is odd and mn is even, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+mnq}}{\prod_{j=0}^m F_{2nk+2jnq}} = \frac{1}{L_{mnq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{2nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+nq}}{F_{2nk} F_{2nk+2nq}} = \frac{1}{L_{nq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{F_{2nk}}. \quad (2.64)$$

Proof. From identity (1.7a) with $v = mnq$ and $u = 2nk + mnq$ comes the identity

$$\frac{L_{mnq} F_{2nk+mnq}}{F_{2nk} F_{2nk+2mnq}} = \frac{1}{F_{2nk}} + \frac{1}{F_{2nk+2mnq}}, \quad mnq \text{ even}. \quad (2.65)$$

From identity (2.65) and Lemma 2.3 with $f(k) = 1/F_{2k}$ we have the finite summation identity:

$$\begin{aligned} L_{mnq} \sum_{k=1}^N \frac{(-1)^{k-1} F_{2nk+mnq}}{\prod_{j=0}^m F_{2nk+2jnq}} &= \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{2nk+2jnq}} \\ &\quad + (-1)^{N-1} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{2nk+2nN+2jnq}}, \end{aligned} \quad (2.66)$$

from which Theorem 2.29 follows in the limit as N approaches infinity. \square

Theorem 2.30. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \frac{L_{2nk+mnq}}{\prod_{j=0}^m F_{2nk+2jnq}} = \frac{1}{F_{mnq}} \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{2nk+2jnq}}.$$

In particular,

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{L_{2nk+nq}}{F_{2nk} F_{2nk+2nq}} &= \frac{1}{F_{nq}} \sum_{k=1}^q \frac{1}{F_{2nk}}, \quad (2.67) \\ \sum_{k=1}^{\infty} \frac{L_{2nk+3nq}}{F_{2nk} F_{2nk+2nq} F_{2nk+4nq} F_{2nk+6nq}} &= \frac{1}{F_{3nq}} \sum_{k=1}^q \frac{1}{F_{2nk} F_{2nk+2nq} F_{2nk+4nq}}. \end{aligned} \quad (2.68)$$

Proof. From identity (1.7b) with $v = mnq$ and $u = 2nk + mnq$ comes the identity

$$\frac{F_{mnq} L_{2nk+mnq}}{F_{2nk} F_{2nk+2mnq}} = \frac{1}{F_{2nk}} - \frac{1}{F_{2nk+2mnq}}, \quad mnq \text{ even}. \quad (2.69)$$

From identity (2.69) and Lemma 2.1 with $f(k) = 1/F_{2k}$ we have the finite summation identity:

$$F_{mnq} \sum_{k=1}^N \frac{L_{2nk+mnq}}{\prod_{j=0}^m F_{2nk+2jnq}} = \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{2nk+2jnq}} - \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{2nk+2nN+2jnq}}, \quad (2.70)$$

from which Theorem 2.30 follows in the limit as N approaches infinity. \square

Theorem 2.31. *If m, n and q are positive integers such that q is even or mnq is odd, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{2nk+mnq}}{\prod_{j=0}^m F_{2nk+2jnq}} = \frac{1}{F_{mnq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{2nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{2nk+nq}}{F_{2nk} F_{2nk+2nq}} = \frac{1}{F_{nq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{F_{2nk}}, \quad q \text{ even or } nq \text{ odd}. \quad (2.71)$$

The alternating summation identity here, valid for q even or mnq odd, is

$$F_{mnq} \sum_{k=1}^N \frac{(-1)^{k-1} L_{2nk+mnq}}{\prod_{j=0}^m F_{2nk+2jnq}} = \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{2nk+2jnq}} + (-1)^{N-1} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{2nk+2nN+2jnq}}. \quad (2.72)$$

2.9 Sums with $L_{2nk} L_{2nk+2nq} L_{2nk+4nq} L_{2nk+6nq} \cdots L_{2nk+2mnq}$ in the denominator

The results in this section are derived from identities (1.9). The proofs are identical to those in section 2.8 and are therefore omitted.

Theorem 2.32. *If m, n and q are positive odd integers, then*

$$\sum_{k=1}^{\infty} \frac{L_{2nk+mnq}}{\prod_{j=0}^m L_{2nk+2jnq}} = \frac{1}{L_{mnq}} \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} L_{2nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{L_{2nk+nq}}{L_{2nk}L_{2nk+2nq}} = \frac{1}{L_{nq}} \sum_{k=1}^q \frac{1}{L_{2nk}}, \quad nq \text{ odd}, \quad (2.73)$$

$$\sum_{k=1}^{\infty} \frac{L_{2nk+3nq}}{L_{2nk}L_{2nk+2nq}L_{2nk+4nq}L_{2nk+6nq}} = \frac{1}{L_{3nq}} \sum_{k=1}^q \frac{1}{L_{2nk}L_{2nk+2nq}L_{2nk+4nq}}. \quad (2.74)$$

The finite summation identity is

$$L_{mnq} \sum_{k=1}^N \frac{L_{2nk+mnq}}{\prod_{j=0}^m L_{2nk+2jnq}} = \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} L_{2nk+2jnq}} \quad mnq \text{ odd} \quad (2.75)$$

$$- \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} L_{2nk+2nN+2jnq}}.$$

Theorem 2.33. *If m , n and q are positive integers such that q is odd and mn is even, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{2nk+mnq}}{\prod_{j=0}^m L_{2nk+2jnq}} = \frac{1}{L_{mnq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{2nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{2nk+nq}}{L_{2nk}L_{2nk+2nq}} = \frac{1}{L_{nq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{L_{2nk}}, \quad q \text{ odd}, n \text{ even}. \quad (2.76)$$

The alternating finite summation identity is

$$L_{mnq} \sum_{k=1}^N \frac{(-1)^{k-1} L_{2nk+mnq}}{\prod_{j=0}^m L_{2nk+2jnq}} = \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{2nk+2jnq}} \quad q \text{ odd}, mn \text{ even}$$

$$+ (-1)^{N-1} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{2nk+2nN+2jnq}}. \quad (2.77)$$

Theorem 2.34. *If m , n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \frac{F_{2nk+mnq}}{\prod_{j=0}^m L_{2nk+2jnq}} = \frac{1}{5F_{mnq}} \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} L_{2nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{F_{2nk+nq}}{L_{2nk}L_{2nk+2nq}} = \frac{1}{5F_{nq}} \sum_{k=1}^q \frac{1}{L_{2nk}}, \quad nq \text{ even}, \quad (2.78)$$

$$\sum_{k=1}^{\infty} \frac{F_{2nk+3nq}}{L_{2nk}L_{2nk+2nq}L_{2nk+4nq}L_{2nk+6nq}} = \frac{1}{5F_{3nq}} \sum_{k=1}^q \frac{1}{L_{2nk}L_{2nk+2nq}L_{2nk+4nq}}. \quad (2.79)$$

The finite summation identity is

$$\begin{aligned} 5F_{mnq} \sum_{k=1}^N \frac{F_{2nk+mnq}}{\prod_{j=0}^m L_{2nk+2jnq}} &= \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} L_{2nk+2jnq}} \quad mnq \text{ even} \\ &\quad - \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} L_{2nk+2nN+2jnq}}, \end{aligned} \quad (2.80)$$

Theorem 2.35. *If m , n and q are positive integers such that q is even or mnq is odd, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+mnq}}{\prod_{j=0}^m L_{2nk+2jnq}} = \frac{1}{5F_{mnq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{2nk+2jnq}}.$$

In particular,

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+nq}}{L_{2nk}F_{2nk+2nq}} = \frac{1}{5F_{nq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{L_{2nk}}, \quad q \text{ even or } nq \text{ odd}. \quad (2.81)$$

The alternating summation identity here, valid for q even or mnq odd, is

$$\begin{aligned} 5F_{mnq} \sum_{k=1}^N \frac{(-1)^{k-1} F_{2nk+mnq}}{\prod_{j=0}^m L_{2nk+2jnq}} &= \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{2nk+2jnq}} \\ &\quad + (-1)^{N-1} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{2nk+2nN+2jnq}}. \end{aligned} \quad (2.82)$$

**2.10 Sums with $F_{nk}^2 F_{nk+nq}^2 \cdots F_{nk+mnq-nq}^2 F_{nk+mnq}^2 F_{nk+mnq+nq}^2 \cdots F_{nk+2mnq}^2$
or $F_{nk}^2 F_{nk+nq}^2 \cdots F_{nk+mnq}^2$ in the denominator**

Theorem 2.36. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{F_{2nk+mnq}}{\prod_{j=0}^m F_{nk+jnq}^2} \right] = \frac{1}{F_{mnq}} \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{nk+jnq}^2}.$$

Explicitly,

$$\sum_{k=1}^{\infty} \frac{F_{2nk+mnq}}{F_{nk}^2 F_{nk+nq}^2 \cdots F_{nk+mnq}^2} = \frac{1}{F_{mnq}} \sum_{k=1}^q \frac{1}{F_{nk}^2 F_{nk+nq}^2 \cdots F_{nk+(m-1)nq}^2}.$$

Examples include:

At $m = 1$

$$\sum_{k=1}^{\infty} \frac{F_{2nk+nq}}{F_{nk}^2 F_{nk+nq}^2} = \frac{1}{F_{nq}} \sum_{k=1}^q \frac{1}{F_{nk}^2}, \quad nq \text{ even}. \quad (2.83)$$

At $(m, n, q) = (1, 1, 2)$ and $(m, n, q) = (1, 2, 1)$:

$$\sum_{k=1}^{\infty} \frac{F_{2k+2}}{F_k^2 F_{k+2}^2} = 2, \quad \sum_{k=1}^{\infty} \frac{F_{4k+2}}{F_{2k}^2 F_{2k+2}^2} = 1. \quad (2.84)$$

At $(m, n, q) = (3, 2, 2)$:

$$\sum_{k=1}^{\infty} \frac{F_{4k+12}}{F_{2k}^2 F_{2k+4}^2 F_{2k+8}^2 F_{2k+12}^2} = \frac{1288981}{35850395750400}. \quad (2.85)$$

Corollary 2.37. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty} \left[\frac{L_{nk+mnq}}{F_{nk+mnq} \prod_{j=0}^{m-1} F_{nk+jnq}^2 \prod_{j=m+1}^{2m} F_{nk+jnq}^2} \right] = \frac{1}{F_{2mnq}} \sum_{k=1}^q \left[\prod_{j=0}^{2m-1} \frac{1}{F_{nk+jnq}^2} \right].$$

Explicitly, Corollary 2.37 is

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{L_{nk+mnq}}{F_{nk}^2 F_{nk+nq}^2 \cdots F_{nk+(m-1)nq}^2 F_{nk+mnq}^2 F_{nk+(m+1)nq}^2 \cdots F_{nk+2mnq}^2} \\ = \frac{1}{F_{2mnq}} \sum_{k=1}^q \frac{1}{F_{nk}^2 F_{nk+nq}^2 \cdots F_{nk+(2m-1)nq}^2}. \end{aligned}$$

Below are a couple of examples:

At $(m, n, q) = (1, 1, 1)$ and $(m, n, q) = (2, 1, 1)$:

$$\sum_{k=1}^{\infty} \frac{L_{k+1}}{F_k^2 F_{k+1} F_{k+2}^2} = 1, \quad \sum_{k=1}^{\infty} \frac{L_{k+2}}{F_k^2 F_{k+1}^2 F_{k+2} F_{k+3}^2 F_{k+4}^2} = \frac{1}{108}. \quad (2.86)$$

At $(m, n, q) = (3, 2, 2)$:

$$\begin{aligned} & \sum_{k=1}^{\infty} \frac{L_{2k+12}}{F_{2k}^2 F_{2k+4}^2 F_{2k+8}^2 F_{2k+12}^2 F_{2k+16}^2 F_{2k+20}^2 F_{2k+24}^2} \\ &= \frac{636693716175181614930457}{1701394375843622618689225675379000792710492054565683200}. \end{aligned} \quad (2.87)$$

Proof of Theorem 2.36

By making appropriate choices for the indices u and v in the identity (1.12a), it is straightforward to establish the following identity:

$$\frac{F_{mnq} F_{2nk+mnq}}{F_{nk}^2 F_{nk+mnq}^2} = \frac{1}{F_{nk}^2} - \frac{(-1)^{mnq}}{F_{nk+mnq}^2}, \quad (2.88)$$

so that

$$\frac{F_{mnq} F_{2nk+mnq}}{F_{nk}^2 F_{nk+mnq}^2} = \frac{1}{F_{nk}^2} - \frac{1}{F_{nk+mnq}^2}, \quad mnq \text{ even} \quad (2.89)$$

and

$$\frac{F_{mnq} F_{2nk+mnq}}{F_{nk}^2 F_{nk+mnq}^2} = \frac{1}{F_{nk}^2} + \frac{1}{F_{nk+mnq}^2}, \quad mnq \text{ odd}. \quad (2.90)$$

From (2.89), with $f(k) = 1/F_k^2$ in Lemma 2.1, we have the finite summation identity

$$\begin{aligned} \sum_{k=1}^N \frac{F_{2nk+mnq}}{\prod_{j=0}^m F_{nk+jnq}^2} &= \frac{1}{F_{mnq}} \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{nk+jnq}^2} \\ &\quad - \frac{1}{F_{mnq}} \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{nk+nN+jnq}^2}, \quad mnq \text{ even}. \end{aligned} \quad (2.91)$$

As N approaches infinity, we have Theorem 2.36, while specifically requiring m to be even gives Corollary 2.37.

Theorem 2.38. *If m , n and q are integers such that q is even or mnq is odd, then*

$$\sum_{k=1}^{\infty} \left[\frac{(-1)^{k-1} F_{2nk+mnq}}{\prod_{j=0}^m F_{nk+jnq}^2} \right] = \frac{1}{F_{mnq}} \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+jnq}^2} \right].$$

Proof. If q is even, then the statement of the theorem follows from (2.89) and identity (2.4) with the upper sign or from (2.90) and identity (2.4) with the lower sign if mnq is odd. \square

In particular, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+nq}}{F_{nk}^2 F_{nk+nq}^2} = \frac{1}{F_{nq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{F_{nk}^2}, \quad q \text{ even or } nq \text{ odd}, \quad (2.92)$$

which generalizes Brousseau's result (Formula (6) of [2], also derived by Melham in reference [8] as a special case of a more general result). Brousseau's formula (6) corresponds to $n = 1$ in (2.92). Brousseau's formula (15) in [2] is also contained in the identity (2.92) above at $n = 3$, $q = 1$.

More examples from Theorem 2.38:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+3nq}}{F_{nk}^2 F_{nk+nq}^2 F_{nk+2nq}^2 F_{nk+3nq}^2} = \frac{1}{F_{3nq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{F_{nk}^2 F_{nk+nq}^2 F_{nk+2nq}^2}, \quad q \text{ even or } nq \text{ odd}. \quad (2.93)$$

Corollary 2.39. *If q is a positive even integer, then*

$$\sum_{k=1}^{\infty} \left[\frac{(-1)^{k-1} L_{nk+mnq}}{F_{nk+mnq} \prod_{j=0}^{m-1} F_{nk+jnq}^2 \prod_{j=m+1}^{2m} F_{nk+jnq}^2} \right] = \frac{1}{F_{2mnq}} \sum_{k=1}^q (-1)^{k-1} \left[\prod_{j=0}^{2m-1} \frac{1}{F_{nk+jnq}^2} \right].$$

In particular:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} L_{k+q}}{F_k^2 F_{k+q} F_{k+2q}^2} = \frac{1}{F_{2q}} \sum_{k=1}^q \frac{(-1)^{k-1}}{F_k^2 F_{k+q}^2}, \quad q \text{ even}. \quad (2.94)$$

We note that Theorem 2.36, Theorem 2.38, Corollary 2.37 and Corollary 2.39 correspond to setting $p = 0$ in Theorem 2.48 and Theorem 2.49 of section 2.12.

Theorem 2.40. *If m , n and q are positive integers, then*

$$\sum_{k=1}^{\infty} \left[\frac{(-1)^{nk-1} F_{2nk+mnq} \prod_{j=1}^{m-1} L_{nk+jnq}^2}{\prod_{j=0}^m F_{nk+jnq}^2} \right] = \frac{5mq}{4F_{mnq}} - \frac{1}{4F_{mnq}} \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{L_{nk+jnq}^2}{F_{nk+jnq}^2} \right].$$

In particular

$$\sum_{k=1}^{\infty} \frac{(-1)^{nk-1} F_{2nk+nq}}{F_{nk}^2 F_{nk+nq}^2} = \frac{5q}{4F_{nq}} - \frac{1}{4F_{nq}} \sum_{k=1}^q \frac{L_{nk}^2}{F_{nk}^2}. \quad (2.95)$$

Theorem 2.41. *If n and q are positive integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{F_{2nk+mnq} \prod_{j=1}^{m-1} L_{nk+jnq}^2}{\prod_{j=0}^m F_{nk+jnq}^2} \right] = \frac{1}{4F_{mnq}} \sum_{k=1}^q \left[(-1)^k \prod_{j=0}^{m-1} \frac{L_{nk+jnq}^2}{F_{nk+jnq}^2} \right].$$

Proof of Theorem 2.40 and Theorem 2.41

Multiplying identity (1.8a) and identity (1.8b) and choosing u and v judiciously, it is easy to establish that the following identity holds for positive integers m , n , q and k :

$$(-1)^{nk-1} \frac{4F_{mnq} F_{2nk+mnq}}{F_{nk}^2 F_{nk+mnq}^2} = \frac{L_{nk+mnq}^2}{F_{nk+mnq}^2} - \frac{L_{nk}^2}{F_{nk}^2}. \quad (2.96)$$

From identity (2.96) and $f(k) = L_k^2/F_k^2$ in Lemma 2.1 we have the finite summation formula:

$$\begin{aligned} & 4F_{mnq} \sum_{k=1}^N \frac{(-1)^{nk-1} F_{2nk+mnq} \prod_{j=1}^{m-1} L_{nk+jnq}^2}{\prod_{j=0}^m F_{nk+jnq}^2} \\ &= \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{L_{nk+nN+jnq}^2}{F_{nk+nN+jnq}^2} - \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{L_{nk+jnq}^2}{F_{nk+jnq}^2}, \end{aligned} \quad (2.97)$$

from which Theorem 2.40 follows in the limit N approaches infinity. Theorem 2.41 follows from identity (2.4).

**2.11 Sums with $L_{nk}^2 L_{nk+nq}^2 \cdots L_{nk+mnq-nq}^2 L_{nk+mnq}^2 L_{nk+mnq+nq}^2 \cdots L_{nk+2mnq}^2$
or $L_{nk}^2 L_{nk+nq}^2 \cdots L_{nk+mnq}^2$ in the denominator**

The theorems in this section are the Lucas versions of those of the previous section. We omit their proofs. The basic identity is

$$\frac{5F_{mnq} F_{2nk+mnq}}{L_{nk}^2 L_{nk+mnq}^2} = \frac{1}{L_{nk}^2} - \frac{(-1)^{mnq}}{L_{nk+mnq}^2}, \quad (2.98)$$

which follows from the identity (1.12b).

Theorem 2.42. *If m, n and q are positive integers such that mnq is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{F_{2nk+mnq}}{\prod_{j=0}^m L_{nk+jnq}^2} \right] = \frac{1}{5F_{mnq}} \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{1}{L_{nk+jnq}^2} \right].$$

Explicitly,

$$\sum_{k=1}^{\infty} \frac{F_{2nk+mnq}}{L_{nk}^2 L_{nk+nq}^2 \cdots L_{nk+mnq}^2} = \frac{1}{5F_{mnq}} \sum_{k=1}^q \frac{1}{L_{nk}^2 L_{nk+nq}^2 \cdots L_{nk+(m-1)nq}^2}.$$

Corollary 2.43. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty} \left[\frac{F_{nk+mnq}}{L_{nk+mnq}^2 \prod_{j=0}^{m-1} L_{nk+jnq}^2 \prod_{j=m+1}^{2m} L_{nk+jnq}^2} \right] = \frac{1}{5F_{2mnq}} \sum_{k=1}^q \left[\prod_{j=0}^{2m-1} \frac{1}{L_{nk+jnq}^2} \right].$$

Theorem 2.44. *If m, n and q are positive integers such that q is even or mnq is odd, then*

$$\sum_{k=1}^{\infty} \left[\frac{(-1)^{k-1} F_{2nk+mnq}}{\prod_{j=0}^m L_{nk+jnq}^2} \right] = \frac{1}{5F_{mnq}} \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+jnq}^2} \right].$$

In particular, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+nq}}{L_{nk}^2 L_{nk+nq}^2} = \frac{1}{5F_{nq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{L_{nk}^2}, \quad q \text{ even or } nq \text{ odd}, \quad (2.99)$$

Corollary 2.45. *If q is a positive even integer, then*

$$\sum_{k=1}^{\infty} \left[\frac{(-1)^{k-1} F_{nk+mnq}}{L_{nk+mnq}^2 \prod_{j=0}^{m-1} L_{nk+jnq}^2 \prod_{j=m+1}^{2m} L_{nk+jnq}^2} \right] = \frac{1}{5F_{2mnq}} \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{2m-1} L_{nk+jnq}^2} \right].$$

In particular:

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{k+q}}{L_k^2 L_{k+q} L_{k+2q}^2} = \frac{1}{5F_{2q}} \sum_{k=1}^q \frac{(-1)^{k-1}}{L_k^2 L_{k+q}^2}, \quad q \text{ even}. \quad (2.100)$$

Theorem 2.46. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty} \left[\frac{(-1)^{nk-1} F_{2nk+mnq} \prod_{j=1}^{m-1} F_{nk+jnq}^2}{\prod_{j=0}^m L_{nk+jnq}^2} \right] = \frac{1}{4F_{mnq}} \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{F_{nk+jnq}^2}{L_{nk+jnq}^2} \right] - \frac{1}{4F_{mnq}} \frac{q}{5^m}.$$

In particular

$$\sum_{k=1}^{\infty} \frac{(-1)^{nk-1} F_{2nk+nq}}{L_{nk}^2 L_{nk+nq}^2} = \frac{1}{4F_{nq}} \sum_{k=1}^q \frac{F_{nk}^2}{L_{nk}^2} - \frac{q}{20F_{nq}}. \quad (2.101)$$

Theorem 2.47. *If n and q are positive integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{F_{2nk+mnq} \prod_{j=1}^{m-1} F_{nk+jnq}^2}{\prod_{j=0}^m L_{nk+jnq}^2} \right] = \frac{1}{4F_{mnq}} \sum_{k=1}^q \left[(-1)^{k-1} \prod_{j=0}^{m-1} \frac{F_{nk+jnq}^2}{L_{nk+jnq}^2} \right].$$

In particular

$$\sum_{k=1}^{\infty} \frac{F_{2nk+nq}}{L_{nk}^2 L_{nk+nq}^2} = \frac{1}{4F_{nq}} \sum_{k=1}^q (-1)^{k-1} \frac{F_{nk}^2}{L_{nk}^2}, \quad n \text{ odd}, q \text{ even}. \quad (2.102)$$

2.12 Sums with $F_{nk} F_{nk+np} F_{nk+nq} F_{nk+nq+np} F_{nk+2nq} F_{nk+2nq+np} \cdots F_{nk+mnq} F_{nk+mnq+np}$ in the denominator

Theorem 2.48. *If m, n, q are positive integers such that mnq is even; and p is a non-negative integer, then*

$$\sum_{k=1}^{\infty} \frac{F_{2nk+mnq+np}}{\prod_{j=0}^m F_{nk+jnq} F_{nk+jnq+np}} = \frac{1}{F_{mnq}} \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} F_{nk+jnq} F_{nk+jnq+np}}.$$

Theorem 2.49. *If m, n, q are positive integers such that mnq is odd or q is even; and p is a non-negative integer, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+mnq+np}}{\prod_{j=0}^m F_{nk+jnq} F_{nk+jnq+np}} = \frac{1}{F_{mnq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} F_{nk+jnq} F_{nk+jnq+np}}.$$

Proof of Theorem 2.48 and Theorem 2.49

By dividing through the identity (1.13a) by $F_u F_{u+p} F_{u+v} F_{u+v+p}$ and setting $u = nk$ and $v = mnq$, the following identity is established

$$\frac{F_{mnq} F_{2nk+mnq+np}}{F_{nk} F_{nk+np} F_{nk+mnq} F_{nk+mnq+np}} = \frac{1}{F_{nk} F_{nk+np}} + \frac{(-1)^{mnq+1}}{F_{nk+mnq} F_{nk+mnq+np}}, \quad (2.103)$$

so that

$$\begin{aligned} & \frac{F_{mnq} F_{2nk+mnq+np}}{F_{nk} F_{nk+np} F_{nk+mnq} F_{nk+mnq+np}} \\ &= \frac{1}{F_{nk} F_{nk+np}} - \frac{1}{F_{nk+mnq} F_{nk+mnq+np}}, \quad mnq \text{ even} \end{aligned} \quad (2.104)$$

and

$$\begin{aligned} & \frac{F_{mnq} F_{2nk+mnq+np}}{F_{nk} F_{nk+np} F_{nk+mnq} F_{nk+mnq+np}} \\ &= \frac{1}{F_{nk} F_{nk+np}} + \frac{1}{F_{nk+mnq} F_{nk+mnq+np}}, \quad mnq \text{ odd}. \end{aligned} \quad (2.105)$$

From (2.104) and $f(k) = 1/(F_k F_{k+np})$ in Lemma 2.1 we obtain the finite summation identity

$$\begin{aligned} & F_{mnq} \sum_{k=1}^N \left[\frac{F_{2nk+mnq+np}}{\prod_{j=0}^m F_{nk+jnq} F_{nk+jnq+np}} \right] \\ &= \sum_{k=1}^q \left[\frac{1}{\prod_{j=0}^{m-1} F_{nk+jnq} F_{nk+jnq+np}} \right] \\ &\quad - \sum_{k=1}^q \left[\frac{1}{\prod_{j=0}^{m-1} F_{nk+nN+jnq} F_{nk+nN+jnq+np}} \right], \end{aligned} \quad (2.106)$$

from which Theorem 2.48 follows in the limit as N approaches infinity. If q is even, then the statement of Theorem 2.49 follows from (2.104) and identity (2.4) with the upper sign or from (2.105) and identity (2.4) with the lower sign if mnq is odd.

2.13 Sums with $L_{nk}L_{nk+np}L_{nk+nq}L_{nk+nq+np}L_{nk+2nq}L_{nk+2nq+np}\cdots$ $L_{nk+mnq}L_{nk+mnq+np}$ in the denominator

The results in this section are the Lucas versions of the results in the preceding section. The basic identities are

$$\frac{5F_{mnq}F_{2nk+mnq+np}}{\prod_{j=0}^m L_{nk+jnq}L_{nk+jnq+np}} = \frac{1}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}} + \frac{(-1)^{mnq+1}}{\prod_{j=1}^m L_{nk+jnq}L_{nk+jnq+np}}, \quad (2.107)$$

and, if mnq is even

$$\begin{aligned} 5F_{mnq} \sum_{k=1}^N \left[\frac{F_{2nk+mnq+np}}{\prod_{j=0}^m L_{nk+jnq}L_{nk+jnq+np}} \right] \\ = \sum_{k=1}^q \left[\frac{1}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}} \right] \\ - \sum_{k=1}^q \left[\frac{1}{\prod_{j=0}^{m-1} L_{nk+nN+jnq}L_{nk+nN+jnq+np}} \right], \end{aligned} \quad (2.108)$$

while if mnq is odd or q is even:

$$\begin{aligned} 5F_{mnq} \sum_{k=1}^N \left[\frac{(-1)^{k-1} F_{2nk+mnq+np}}{\prod_{j=0}^m L_{nk+jnq}L_{nk+jnq+np}} \right] \\ = \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}} \right] \\ + (-1)^{N-1} \sum_{k=1}^q \left[\frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+nN+jnq}L_{nk+nN+jnq+np}} \right]. \end{aligned} \quad (2.109)$$

Theorem 2.50. *If m, n, q are positive integers such that mnq is even; and p is a non-negative integer, then*

$$\sum_{k=1}^{\infty} \frac{F_{2nk+mnq+np}}{\prod_{j=0}^m L_{nk+jnq}L_{nk+jnq+np}} = \frac{1}{5F_{mnq}} \sum_{k=1}^q \frac{1}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}}.$$

Theorem 2.51. *If m, n, q are positive integers such that mnq is odd or q is even; and p is a non-negative integer, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+mnq+np}}{\prod_{j=0}^m L_{nk+jnq}L_{nk+jnq+np}} = \frac{1}{5F_{mnq}} \sum_{k=1}^q \frac{(-1)^{k-1}}{\prod_{j=0}^{m-1} L_{nk+jnq}L_{nk+jnq+np}}.$$

2.14 Evaluation of other sums

Theorem 2.52. *If m, n and q are positive integers, then*

$$\sum_{k=1}^{\infty} \frac{(-1)^{nk-1} F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^m F_{nk+jnq} F_{nk+jnq+2}} = \frac{q}{F_{mnq}} - \frac{1}{F_{mnq}} \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}},$$

so that

$$\sum_{k=1}^{\infty} \frac{F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^m F_{nk+jnq} F_{nk+jnq+2}} = \frac{1}{F_{mnq}} \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}} - \frac{q}{F_{mnq}}, \quad n \text{ even} \quad (2.110)$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1} F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^m F_{nk+jnq} F_{nk+jnq+2}} = \frac{q}{F_{mnq}} - \frac{1}{F_{mnq}} \sum_{k=1}^q \prod_{j=0}^{m-1} \frac{F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}}, \quad n \text{ odd}. \quad (2.111)$$

Theorem 2.53. *If m, n and q are positive integers such that n is odd and q is even, then*

$$\sum_{k=1}^{\infty} \left[\frac{F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^m F_{nk+jnq} F_{nk+jnq+2}} \right] = \frac{1}{F_{mnq}} \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{(-1)^k F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}} \right].$$

Proof of Theorem (2.52) and Theorem (2.53)

Dividing through the identity (1.10) by $F_{v+1}F_{v-1}F_{u+1}F_{u-1}$ and setting $u = nk + 1$ and $v = nk + mnq + 1$ we obtain the identity

$$\frac{(-1)^{nk-1} F_{mnq} F_{2nk+mnq+2}}{F_{nk} F_{nk+2} F_{nk+mnq} F_{nk+mnq+2}} = \frac{F_{nk+mnq+1}^2}{F_{nk+mnq+2} F_{nk+mnq}} - \frac{F_{nk+1}^2}{F_{nk+2} F_{nk}}. \quad (2.112)$$

With $f(k) = F_{k+1}^2/(F_k F_{k+2})$ in Lemma 2.1 and use of the identity (2.112) we get the finite summation identity

$$\begin{aligned}
F_{mnq} \sum_{k=1}^N & \left[\frac{(-1)^{nk-1} F_{2nk+mnq+2} \prod_{j=1}^{m-1} F_{nk+jnq+1}^2}{\prod_{j=0}^m F_{nk+jnq} F_{nk+jnq+2}} \right] \\
&= \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{F_{nk+nN+jnq+1}^2}{F_{nk+nN+jnq} F_{nk+nN+jnq+2}} \right] \\
&\quad - \sum_{k=1}^q \left[\prod_{j=0}^{m-1} \frac{F_{nk+jnq+1}^2}{F_{nk+jnq} F_{nk+jnq+2}} \right],
\end{aligned} \tag{2.113}$$

from which Theorem (2.52) follows in the limit as N approaches infinity.

Theorem (2.53) follows from the identity (2.112) and taking $f(k) = F_{k+1}^2/(F_k F_{k+2})$ in identity (2.4) with the upper sign.

Theorem 2.54.

$$\sum_{k=1}^{\infty} \frac{F_{2k+3}}{F_k^4 F_{k+1}^3 F_{k+2}^3 F_{k+3}^4} = \frac{1}{128}, \quad \sum_{k=1}^{\infty} \frac{F_{2k+3}}{L_k^4 L_{k+1}^3 L_{k+2}^3 L_{k+3}^4} = \frac{1}{829440}.$$

Theorem 2.55.

$$\sum_{k=1}^{\infty} \frac{F_{3k+1} F_{3k+2} F_{6k+3}}{F_{3k}^4 F_{3k+3}^4} = \frac{1}{128}, \quad \sum_{k=1}^{\infty} \frac{L_{3k+1} L_{3k+2} F_{6k+3}}{L_{3k}^4 L_{3k+3}^4} = \frac{1}{10240}.$$

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